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# Phase Change Simulation for Isothermal Compressible Two-Phase Flows

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**We present a numerical scheme based on a two-step convection-relaxation strategy for the simulation of compressible two-phase flows with phase change. The core system used here is a simple isothermal model where stiff source terms account for mass transfer.**

## I. Introduction

For the past years a great deal of attention has been paid to the simulation of two-phase flows. These flows are involved in a wide range of industrial applications from boiling crisis prediction to cavitating flows. We focus here on the simulation of dynamical phase change between two compressible fluids. Previous works such as Le Metayer[10] managed to achieve such task for the case of very fast mass transfer phenomena. We rather deal here with slower processes as Jamet[8] where both fluids are close to the thermodynamical equilibrium. Moreover, we restrict ourselves to the case of isothermal flows and we also assume that both fluids are in thermal equilibrium.

The phase change model adopted here has been introduced by Caro.[4] This model encompasses two relaxation mechanisms through stiff source terms: a mechanical relaxation effect and a thermodynamical effect that accounts for mass transfer.

We will first briefly recall the basic model and its properties. We shall then describe two equilibrium submodels issued from the relaxations processes when instantaneous equilibrium is assumed. Two numerical schemes will then be introduced. Both schemes are based on a two-step strategy involving a convective step (by the mean of a Godunov-type method) followed by a relaxation step. Finally, we shall present 1D and 2D numerical results with convergence tests.

## II. Governing Equations

We consider a flow involving two phases represented by two compressible fluids. Each fluid $\alpha = 1, 2$ is equipped with an equation of state (EOS) $\rho_\alpha \mapsto f_\alpha(\rho_\alpha)$ verifying

$$P_\alpha = -\rho_\alpha^2 \mathrm{d}f_\alpha/\mathrm{d}\rho_\alpha, \quad \mathrm{d}P_\alpha/\mathrm{d}\rho_\alpha = \rho_\alpha \mathrm{d}g_\alpha/\mathrm{d}\rho_\alpha,$$

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where $\rho_\alpha$ is the density of fluid $\alpha$, $f_\alpha$, $P_\alpha$ and $g_\alpha = f_\alpha + P_\alpha/\rho_\alpha$ are respectively the free energy, the pressure and the free enthalpy of fluid $\alpha$. We note $z_\alpha$, the volume fraction of phase $\alpha$. We suppose that $z_1 + z_2 = 1$ and note $z = z_1$. Setting $m_\alpha = z_\alpha \rho_\alpha$, the model we study here reads

$$\left\{\begin{array}{lcl} \partial_t m_1 + \mathrm{div}(m_2 \mathbf{u}) &=& \lambda(g_2 - g_1), \\ \partial_t m_2 + \mathrm{div}(m_2 \mathbf{u}) &=& -\lambda(g_2 - g_1), \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u} + P\mathrm{Id}) &=& 0, \\ \partial_t z + \mathbf{u} \cdot \nabla z &=& \kappa(P_1 - P_2), \end{array}\right. \tag{1}$$

where $\lambda$ and $\kappa$ are non negative parameters, $\rho = m_1 + m_2$ is the global density and $P$ is defined as $P = \sum_\alpha z_\alpha P_\alpha$. This model shall be referred to as the relaxed system.

### A. Hyperbolicity of the Relaxed System

Considering smooth solutions $\mathbf{V} = (\rho_1 z_1, \rho_2 z_2, \rho u, z)^T$, the system (1) is equivalent in 1D to the quasilinear formulation

$$\partial_t \mathbf{V} + A(\mathbf{V})\partial_x \mathbf{V} = R(\mathbf{V}), \quad R(\mathbf{V}) = \begin{bmatrix} \lambda(g_2 - g_1) \\ -\lambda(g_2 - g_1) \\ 0 \\ \kappa(P_1 - P_2) \end{bmatrix}, \quad A(\mathbf{V}) = \begin{pmatrix} u y_2 & -u y_1 & y_1 & 0 \\ -u y_2 & u y_1 & y_2 & 0 \\ c_1^2 - u^2 & c_2^2 - u^2 & 2u & M \\ 0 & 0 & 0 & u \end{pmatrix},$$

where $y_\alpha = \rho_\alpha z_\alpha / \rho$ is the mass fraction of the $\alpha$ phase, $c_\alpha^2 = \mathrm{d}P_\alpha/\mathrm{d}\rho_\alpha$ is the squared sound velocity of phase $\alpha$ and $M = \partial P/\partial z$. Denoting by $c$ the mixture sound velocity defined by $c^2 = \sum_\alpha y_\alpha c_\alpha^2$, the matrix $A(\mathbf{V})$ possesses three distinct eigenvalues: $\lambda_1 = u - c$, $\lambda_2 = \lambda_3 = u$, $\lambda_4 = u + c$. The corresponding left eigenvectors $\mathbf{l}_k$ and right eigenvectors $\mathbf{r}_k$ read

$$\mathbf{l}_1 = \frac{1}{2c^2}\begin{bmatrix} c_1^2 + uc \\ c_2^2 + uc \\ -c \\ M \end{bmatrix}, \quad \mathbf{l}_2 = \frac{1}{c^2}\begin{bmatrix} -y_1 c_1^2 + c^2 \\ -y_1 c_2^2 \\ 0 \\ -y_1 M \end{bmatrix}, \quad \mathbf{l}_3 = \frac{1}{c^2}\begin{bmatrix} -y_2 c_1^2 \\ c^2 - y_2 c_2^2 \\ 0 \\ -y_2 M \end{bmatrix}, \quad \mathbf{l}_4 = \frac{1}{2c^2}\begin{bmatrix} c_1^2 - uc \\ c_2^2 - uc \\ c \\ M \end{bmatrix},$$

$$\mathbf{r}_1 = \begin{bmatrix} y_1 \\ y_2 \\ u - c \\ 0 \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} 1 \\ 0 \\ u \\ -c_1^2/M \end{bmatrix}, \qquad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 1 \\ u \\ c_2^2/M \end{bmatrix}, \qquad \mathbf{r}_4 = \begin{bmatrix} y_1 \\ y_2 \\ u + c \\ 0 \end{bmatrix}.$$

This eigenstructure shows that the system (1) is hyperbolic.

The system (1) is endowed with a (mathematical) entropy-entropy flux pair $[\rho f + \rho|\mathbf{u}|^2/2, (\rho f + \rho|\mathbf{u}|^2/2 + P)\mathbf{u}]$. Although $(\rho_1 z_1, \rho_2 z_2, \rho \mathbf{u}, z) \mapsto (\rho f + \rho|\mathbf{u}|^2/2)$ is not convex, the following entropy equation is available

$$\partial_t \left(\rho f + \rho \frac{|\mathbf{u}|^2}{2}\right) + \mathrm{div}\left[\left(\rho f + \rho \frac{|\mathbf{u}|^2}{2} + P\right)\mathbf{u}\right] = -\lambda(g_1 - g_2)^2 - \kappa(P_1 - P_2)^2,$$

which allows to derive an entropy inequality associated with the entropy-entropy flux pair

$$\partial_t \left(\rho f + \rho \frac{|\mathbf{u}|^2}{2}\right) + \mathrm{div}\left[\left(\rho f + \rho \frac{|\mathbf{u}|^2}{2} + P\right)\mathbf{u}\right] \leq 0.$$

### B. Limit System $\kappa = +\infty$, $\lambda < +\infty$ (M-equilibrium)

The $\kappa = +\infty$ assumption in the system (1) is formally equivalent to compute $z \in [0,1]$ such that

$$P_1\left(\frac{m_1}{z}\right) = P_2\left(\frac{m_2}{1-z}\right), \tag{2}$$

for given fixed values $m_1 = \rho_1 z_1$ and $m_2 = \rho_2 z_2$.

**Proposition 1.** *We suppose that the functions $P_\alpha : \rho_\alpha \mapsto P_\alpha(\rho_\alpha)$ are $\mathcal{C}^1(\mathbb{R}^+)$, are strictly increasing on $\mathbb{R}^+$ and tend to $+\infty$ when $\rho_\alpha \to +\infty$. Then there exists a unique $z^*(m_1, m_2) \in [0,1]$ solution of the equation (2). Moreover $m_1 \mapsto z^*(m_1, m_2)$ is non-decreasing on $\mathbb{R}^+$, $m_2 \mapsto z^*(m_1, m_2)$ is non-increasing on $\mathbb{R}^+$, $(m_1, m_2) \mapsto z^*(m_1, m_2)$ is as regular as $\rho_\alpha \mapsto P_\alpha$, $\alpha = 1, 2$.*

Using $z^*$, the M-equilibrium system reads

$$\left\{\begin{array}{lcl} \partial_t m_1 + \mathrm{div}(m_1 \mathbf{u}) & = & \lambda(g_2 - g_1), \\ \partial_t m_2 + \mathrm{div}(m_2 \mathbf{u}) & = & -\lambda(g_2 - g_1), \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u} + P\mathrm{Id}) & = & 0, \end{array}\right. \tag{3}$$

with $P = \sum_\alpha z_\alpha^* P_\alpha = P_1 = P_2$.

Considering smooth solutions $\mathbf{W} = (\rho_1 z_1, \rho_2 z_2, \rho u)^T$ of (3), the system (3) is equivalent in 1D to

$$\partial_t \mathbf{W} + B(\mathbf{W})\partial_x \mathbf{W} = \mathcal{S}, \quad B(\mathbf{W}) = \begin{bmatrix} u & 0 & \rho_1 z_1 \\ 0 & u & \rho_2 z_2 \\ (1/\rho)\partial_{m_1} P & (1/\rho)\partial_{m_2} P & u \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} \lambda(g_2 - g_1) \\ -\lambda(g_2 - g_1) \\ 0 \end{bmatrix}.$$

The matrix $B(\mathbf{W})$ possesses three disctinct eigenvalues: $\lambda_1 = u - c^*$, $\lambda_2 = u$, $\lambda_3 = u + c^*$, where $c^*$ is defined by

$$\frac{1}{\rho (c^*)^2} = \sum_\alpha \frac{(z_\alpha^*)^2}{m_\alpha c_\alpha^2}.$$

This ensures the strict hyperbolicity of the M-equilibrium system (3).

### C. Limit System $\kappa = +\infty$, $\lambda = +\infty$ (MT-equilibrium)

The condition $\kappa = +\infty$, $\lambda = +\infty$ (MT-equilibrium) is formally equivalent to determine $m_1$, $m_2$ and $z$, for a given $\rho$, such that

$$P_1\left(\frac{m_1}{z}\right) = P_2\left(\frac{m_2}{1-z}\right), \quad g_1\left(\frac{m_1}{z}\right) = g_2\left(\frac{m_2}{1-z}\right), \quad \rho = m_1 + m_2. \tag{4}$$

System (4) is equivalent to find $\rho_1 > 0$, $\rho_2 > 0$ and $z \in (0,1)$ such that for a given $\rho > 0$

$$P_1(\rho_1) = P_2(\rho_2), \quad g_1(\rho_1) = g_2(\rho_2), \quad \rho = z\rho_1 + (1-z)\rho_2. \tag{5}$$

Let us note that the above system does no longer depend on $m_1$, $m_2$ and $z$, but solely on the parameter $\rho$. Unfortunately, on the contrary of system (2), nor existence nor unicity for the solution of (5) is ensured. However, this issue seems natural. Indeed, it expresses the fact that it is not possible to prescribe a mass transfer mechanism between two fluids with two arbitrary EOS. A certain degree of compatibility between both fluids EOS is required in order to define correctly the thermodynamical equilibrium. This matter can be illustrated in the case of stiffened gas EOS. For such fluids we have

$$P_\alpha = a_\alpha \rho_\alpha - P_\alpha^\infty, \quad g_\alpha = b_\alpha + a_\alpha \log(\rho_\alpha),$$

where the constants $a_\alpha$ and $b_\alpha$ are related to the constant flow temperature $T$ as follows

$$a_\alpha(T) = (\gamma_\alpha - 1)c_{v_\alpha}T, \quad b_\alpha(T) = q_\alpha + \Big(\gamma_\alpha c_{v_\alpha} - q'_\alpha + (\gamma_\alpha - 1)c_{v_\alpha}\log\left[(\gamma_\alpha - 1)c_{v_\alpha}\right] - c_{v_\alpha}\log T\Big)T,$$

the parameters $\gamma_\alpha > 1$, $P_\alpha^\infty \geq 0$, $c_{v\alpha} > 0$, $q_\alpha$ and $q'_\alpha$ being constant for each fluid $\alpha = 1, 2$. In this particular case we have the following solvability condition

**Proposition 2.** *For given $m_1$, $m_2$ and $z$, $\rho_1^*$ and $\rho_2^*$ solutions of (5) verify*

$$\rho_2^* = (\rho_1^*)^{a_1/a_2}\exp\left(\frac{b_1 - b_2}{a_2}\right) \quad \textit{and} \quad a_1\rho_1^* + (P_1^\infty - P_2^\infty) - a_2(\rho_1^*)^{a_1/a_2}\exp\left(\frac{b_1 - b_2}{a_2}\right) = 0. \tag{6}$$

*If $(a_1 - a_2)(P_2^\infty - P_1^\infty) \geq 0$ then system (6) admits a unique solution, while if $(a_1 - a_2)(P_2^\infty - P_1^\infty) < 0$ it has two solutions or no solution.*

**Remark 1.** *For the specific case of perfect gas EOS, system (5) admits a unique explicit solution which reads*

$$\rho_1^* = \exp\left(-\frac{b_1 - b_2}{a_1 - a_2}\right)\left(\frac{a_1}{a_2}\right)^{\frac{a_2}{a_1 - a_2}}, \quad \rho_2^* = \exp\left(-\frac{b_1 - b_2}{a_1 - a_2}\right)\left(\frac{a_1}{a_2}\right)^{\frac{a_1}{a_1 - a_2}}.$$

In the sequel we shall consider that the system (5) possesses at most a unique solution $\rho_1^* > 0$, $\rho_2^* > 0$ and $z^* \in (0, 1)$ for a given $\rho > 0$. Consequently $\rho_1^*$ and $\rho_2^*$ have fixed values which only depend on the fluids EOS definition and, supposing that $\rho_1^* < \rho_2^*$, $z^*$ depends on $\rho$ as follows

$$z^* = 1 \quad \text{if } \rho < \rho_1^*, \qquad z^* = \frac{\rho - \rho_2^*}{\rho_1^* - \rho_2^*} \quad \text{if } \rho_1^* \leq \rho \leq \rho_2^*, \qquad z^* = 0 \quad \text{if } \rho_2^* < \rho. \tag{7}$$

Finally, the MT-equilibrium system is formally equivalent to

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \quad \partial_t\rho\mathbf{u} + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u} + P(\rho)\mathrm{Id}) = 0, \tag{8}$$

where the pressure $P$ is determined by

$$P(\rho) = P_1(\rho) \quad \text{if } z^* = 1, \quad P(\rho) = z^*P_1(\rho_2^*) + (1 - z^*)P_2(\rho_2^*) \quad \text{if } z^* \in (0, 1), \quad P(\rho) = P_2(\rho) \quad \text{if } z^* = 0.$$

The pressure law $\rho \mapsto P(\rho)$ is continuously differentiable except in points $\rho_1^*$ and $\rho_2^*$ where it is only continuous. System (8) has a structure close to the isothermal Euler equation, however the system is only weakly hyperbolic. Indeed, for $\rho \notin \{\rho_1^*, \rho_2^*\}$ it is possible to compute the Jacobian associated with the system. It possesses two eigenvalues $\lambda_1 = u + c$ and $\lambda_2 = u - c$, where $c^2 = \mathrm{d}P/\mathrm{d}\rho$. While $c > 0$ for $\rho \notin [\rho_1^*, \rho_2^*]$, in the case $\rho \in (\rho_1^*, \rho_2^*)$, we have $c = 0$. In this case both eigenvalues collapse and the system looses its eigenvector basis.

**Remark 2.** *The MT-equilibrium system is endowed with a volumic specific free energy $F$ which has the following properties: $\rho \mapsto F(\rho)$ is only weak-convex, pressure $P$ is related to $F$ by $P = \rho\mathrm{d}F/\mathrm{d}\rho - F$ and the graph of $F$ in the $(\rho, F)$ is the convex hull of the function $\rho \mapsto \min[F_1(\rho), F_2(\rho)]$, where $F_\alpha$ is the volumic free energy of the fluid $\alpha$.*

**Remark 3.** *The solution of the Riemann problem for the MT-equilibrium system may not be unique in the class of entropic solutions. In this sense, the sytem is ill-posed.*

## III. Numerical Treatment

From now on, we shall suppose that both fluids are governed by a stiffened gas EOS. Consider $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $\Delta x > 0$ and $\Delta t > 0$ be the space and time steps. Recalling that $\mathbf{W} = (\rho_1 z_1, \rho_2 z_2, \rho u)^T$, we note $\mathbf{W}_i^n$ the finite volume approximate value of $(1/\Delta x)\int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} \mathbf{W}(x, t_n)\mathrm{d}x$ where $t_n = n\Delta t$. The numerical strategy we adopt here is a two-step scheme which consists in a first convective step thanks to a discretization of the relaxed system (1) without source terms

$$\mathbf{W}_i^{n+1/2} - \mathbf{W}_i^n + \frac{\Delta t}{\Delta x}\Big[\mathbf{F}_{i+1/2}(\mathbf{W}^n, z^n) - \mathbf{F}_{i-1/2}(\mathbf{W}^n, z^n)\Big] = 0, \quad z_i^{n+1/2} - z_i^n + \frac{\Delta t}{\Delta x} H_i(\mathbf{W}^n, z^n) = 0, \quad (9)$$

followed by a projection (or relaxation) step onto the equilibrium states

$$(\mathbf{W}^{n+1}, z^{n+1}) = \Pi(\mathbf{W}^{n+1/2}, z^{n+1/2}).$$

The convection step (9) is achieved in similar lines as Chanteperdrix[5] and Allaire[2] using here a Roe-type linearization for $\mathbf{F}_{i+1/2}$ and an upwind scheme for $H_i(\mathbf{W}^n, z^n)$. For the relaxation step, we propose two variants depending on the chosen equilibrium states set.

### A. Relaxation Towards The M-Equilibrium (M-scheme)

For the M-equilibrium, we suppose $\lambda$ to be a finite but very large parameter. We formally set $\kappa = +\infty$ which imposes that the equilibrium value $z^*$ is retrieved by solving equation (2). With the stiffened gas assumption we have

$$z^* = \frac{\beta}{1+\beta}, \quad \beta = \frac{-\Big(a_2 m_2 - a_1 m_1 - P_2^\infty + P_1^\infty\Big) + \sqrt{\Big(a_2 m_2 - a_1 m_1 - P_2^\infty + P_1^\infty\Big)^2 + 4 a_1 a_2 m_1 m_2}}{2 a_2 m_2}. \quad (10)$$

Using a splitting operator method, we integrate for $t \in [0, \Delta t]$ the following ODE system

$$\partial_t m_1 = \lambda\Big[g_2(\rho_2) - g_1(\rho_1)\Big], \quad \partial_t \rho = 0, \quad \partial_t(\rho u) = 0, \quad (\mathbf{W}, z)(0) = (\mathbf{W}, z)^{n+1/2}. \quad (11)$$

For stability purpose, instead of a direct numerical integration of system (11) we seek for an approximate ODE system that allows explicit integration. In order to do so we scale the parameter $\lambda$ by setting $\lambda = m_1 m_2 \tilde{\lambda}$, where $\tilde{\lambda}$ is a supposedly large constant parameter. Then, we also freeze the $(g_2 - g_1)$ term in its value $(g_2 - g_1)^{n+1/2}$ evaluated at $(\mathbf{W}, z)^{n+1/2}$. We obtain the simplified ODE system

$$\partial_t(m_1) = \tilde{\lambda} m_1 (\rho_i^{n+1/2} - m_1)(g_2 - g_1)_i^{n+1/2}, \quad \partial_t \rho = 0, \quad \partial_t(\rho u) = 0. \quad (12)$$

This system can be solved explicitely

$$m_1(t) = \rho_i^{n+1/2}\Big(\Big[\frac{\rho_i^{n+1/2}}{(m_1)_i^{n+1/2}} - 1\Big]\exp[\tilde{\lambda}\rho_i^{n+1/2}(g_2 - g_1)_i^{n+1/2} t] + 1\Big)^{-1}, \ \rho(t) = \rho_i^{n+1/2} \text{ and } u(t) = u_i^{n+1/2}.$$

Let us notice that the above expression ensures $m_1(t) > 0$. We can now complete the source term integration step by setting

$$\rho_i^{n+1} = \rho_i^{n+1/2}, \qquad u_i^{n+1} = u_i^{n+1/2}, \qquad (m_1)_i^{n+1} = m_1(\Delta t, ), \qquad (m_2)_i^{n+1} = \rho_i^{n+1} - (m_1)_i^{n+1}.$$

Finally we update $z$ by remapping $z$ onto the M-equilibrium states by using expression (10) with $\mathbf{W}^{n+1}$.

### B. Relaxation Towards The MT-Equilibrium (MT-scheme)

We suppose the equilibrium values $\rho_1^*$ and $\rho_2^*$ known either explicitly for the case of perfect gases (see remark 1), either by solving (6) with an iterative algorithm for the case of stiffened gas.

The intermediary solution $(\mathbf{W}, z)^{n+1/2}$ obtained after the convective step (9) is projected onto MT-equilibrium states using relation (7). This leads to

$$\begin{array}{llll}
\text{if } \rho_i^{n+1/2} < \rho_1^* & z_i^{n+1} = 1, & (m_1)_i^{n+1} = \rho_i^{n+1/2}, & (m_2)_i^{n+1} = 0, \\
\text{if } \rho_1^* < \rho_i^{n+1/2} < \rho_2^* & z_i^{n+1} = \dfrac{\rho_i^{n+1/2} - \rho_2^*}{\rho_1^* - \rho_2^*}, & (m_1)_i^{n+1} = \rho_1^* z_i^{n+1}, & (m_2)_i^{n+1} = \rho_2^*(1 - z_i^{n+1}), \\
\text{if } \rho_2^* < \rho_i^{n+1/2} & z_i^{n+1} = 0, & (m_1)_i^{n+1} = 0, & (m_2)_i^{n+1} = \rho_i^{n+1/2},
\end{array}$$

and $\rho_i^{n+1} = \rho_i^{n+1/2}$, $u_i^{n+1} = u_i^{n+1/2}$.

## IV. Numerical Results

### A. A One Dimensional Isothermal Phase Change Test

We consider a 1 m long domain containing liquid and vapor initially separated by an interface located at $x = 0.5$ m. We suppose that both phases are perfect gas (characteristics parameters are given in table 1). The temperature is fixed at $T = 100$ K. The fluids are supposed to be initially at rest, the densities are chosen so that pressure is constant in the whole domain: $\rho = \rho_1^* \simeq 6.05921 \times 10^{-4}$ kg.m$^{-3}$ at the left of the interface and $\rho = \rho_2^* \simeq 33.21726 \times 10^{-4}$ kg.m$^{-3}$ at the right.

We suppose that the left boundary is a piston which moves towards left at constant speed $u_p = -100$ m.s$^{-1}$. The right boundary is a reflecting wall. The piston motion will generate an accoustic wave travelling from left to right that will reach the interface. This will perturb the thermodynamical equilbrium of the interface and trigger the system relaxation. A phase change process occurs.

| | left | right |
|---|---|---|
| $C_v$ (J.kg$^{-1}$.K$^{-1}$) | 1 816 | 1 040 |
| $\gamma = C_p/C_v$ | 2.35 | 1.43 |

**Table 1. Fluids parameters for the one dimensional piston test.**

We use here the MT-scheme with different space steps. Figure 1 displays the profiles of $z$, pressure $P$ and velocity $u$ at instants $t = 6.66$ ms and $t = 20$ ms. The solid lines represent the approximate solution with 5000 cells, the $\times$ symbol lines represent the solution with 1000 cells and the $\bigtriangledown$ symbol lines represent the solution with 100 cells. This allows to check the good convergence behaviour of the solution when the space step goes to zero. Let us note that some small pressure oscillations appear for the rough mesh, nevertheless their amplitude decrease while refining the space step.

We now turn to the convergence of the solution obtained with the M-scheme for large values of $\tilde{\lambda}$ to MT-scheme appromixate solution. Figure 2 shows the profiles computed with both M-scheme and MT-schemes for the variable $z$ and the pressure in the case of the piston test. We use a $\tilde{\lambda}$ values ranging from 10 to $10^6$. While for low values of $\tilde{\lambda}$ the solutions are quite different, we see that as expected for $\lambda \to \infty$ they tend to the same profile. Le us consider the MT-scheme computed solution $\mathbf{V}_h^{\text{ref}}$ as reference approximate solution. Let $\mathbf{V}_h^{\tilde{\lambda}}$ be the solution obtained with the M-scheme. We use here a fixed 500 cells mesh. Figure 3 displays the graph of the function $\ln(\tilde{\lambda}) \mapsto \ln\left(\|\mathbf{V}_h^{\text{ref}} - \mathbf{V}_h^{\tilde{\lambda}}\|_{L^1}\right)$. We numerically check that for $\tilde{\lambda} \to +\infty$ we have $\|\mathbf{V}_h^{\tilde{\lambda}} - \mathbf{V}_h^{ref}\|_{L^1} = \mathcal{O}\left(1/\tilde{\lambda}\right)$.

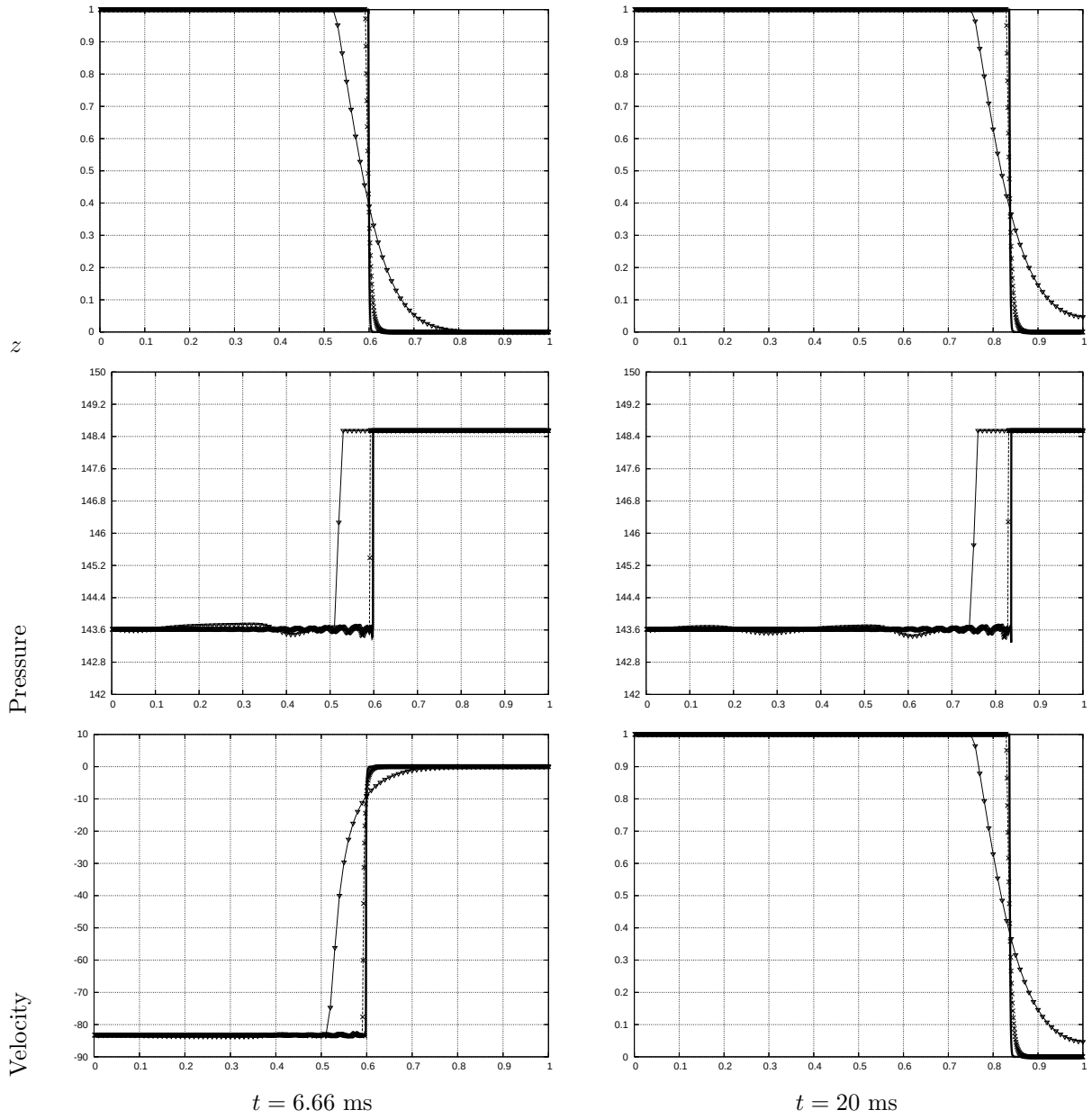


**Figure 1. Profiles of $z$, the pressure of the velocity obtained with the MT-scheme. The solid lines represent the approximate solution with 5000 cells, the $\times$ symbol lines represent the solution with 1000 cells and the $\bigtriangledown$ symbol lines represent the solution with 100 cells.**

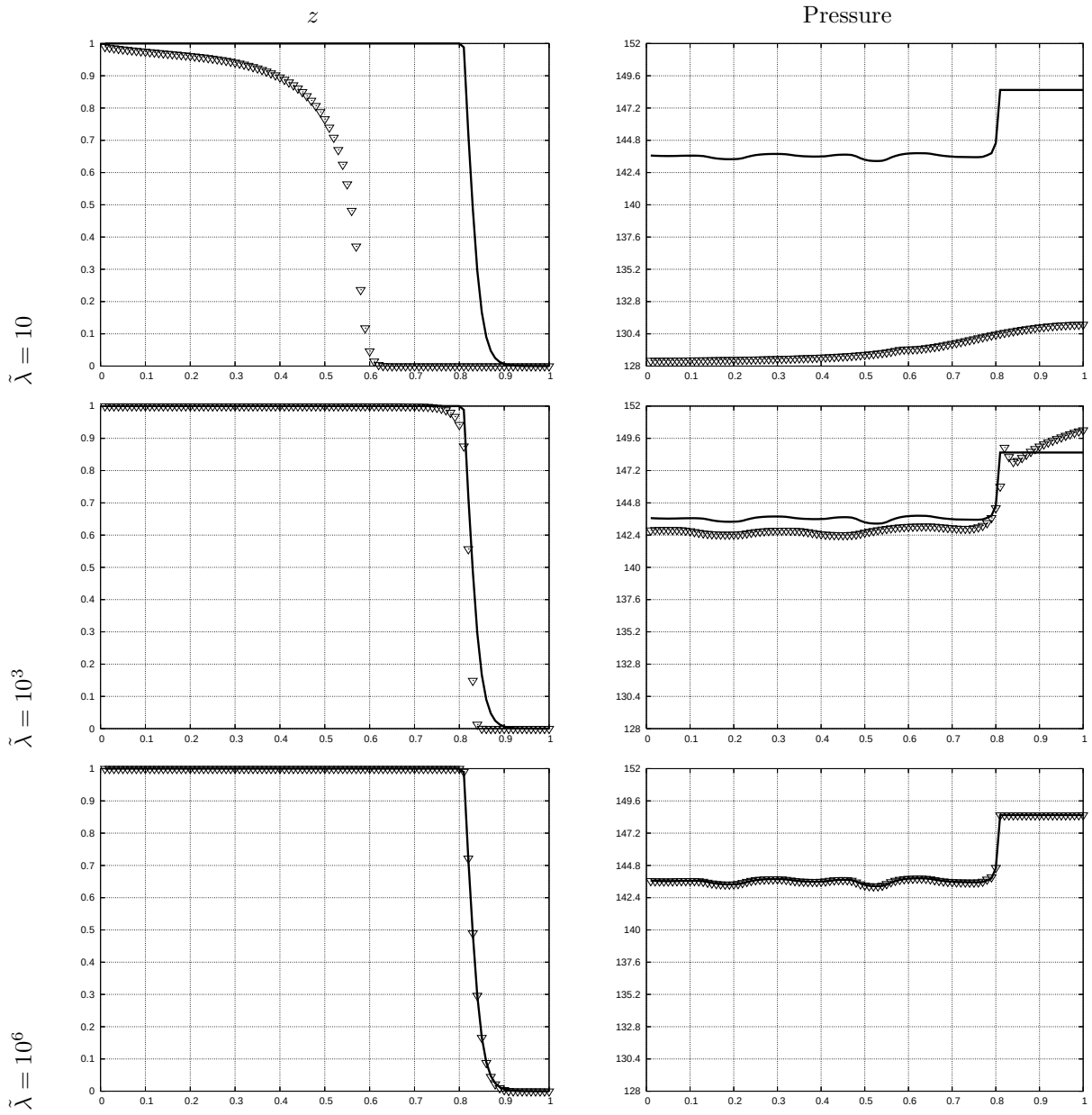


**Figure 2. Profiles of $z$ and the pressure at $t = 20$ ms obtained with the MT-scheme (solid lines) and the M-scheme ($\bigtriangledown$ symbol) for $\tilde{\lambda}$ ranging from 10 to $10^6$.**

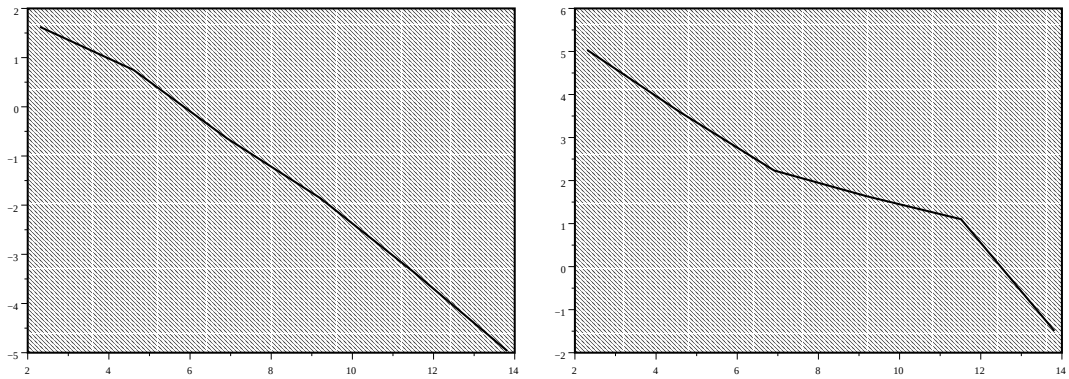

**Figure 3. Convergence behavior for $\lambda \to +\infty$ of the M-equilibrium scheme. The left and right graphs represent respectively $\ln(\tilde{\lambda}) \mapsto \ln\left(\|z_h^{\text{ref}} - z_h^{\tilde{\lambda}}\|_{L^1}\right)$ and $\ln(\tilde{\lambda}) \mapsto \ln\left(\|P_h^{\text{ref}} - P_h^{\tilde{\lambda}}\|_{L^1}\right)$ at $t = 20$ ms.**

### B. Compression of a Gas Bubble

We consider 1 m long square domain discretized over a $100 \times 100$ cells mesh. A gas bubble is surrounded by liquid in the center of the domain. The radius of the gas bubble is initially $r = 25$ cm. The EOS used are stiffened gas type whose coefficients are given in table 2. The temperature is set to $T = 600$ K. The densities of vapor and liquid are obtained by solving the equation (5), which results the following approximate values $\rho = \rho_1^* \simeq 35.56506$ kg.m$^{-3}$ for the vapor and $\rho = \rho_2^* \simeq 686.24091$ kg.m$^{-3}$ for the liquid.

| | $c_v$ (J.kg$^{-1}$.K$^{-1}$) | $\gamma = c_p/c_v$ | $P^\infty$ (Pa) | $q$ (kJ.kg$^{-1}$) | $q'$ (kJ.kg$^{-1}$.K$^{-1}$) |
|---|---|---|---|---|---|
| vapor (bubble) | 1040.14 | 1.43 | 0 | 2 030.255 | −23.31 |
| liquid | 1816.2 | 2.35 | $10^9$ | −1 167.056 | 0 |

**Table 2. Fluids parameters for the two dimensional bubble compression.**

We suppose again the left boundary to be a piston which moves towards left at constant speed $u_p = -100$ m.s$^{-1}$. The other boundaries are reflecting wall. The figure 4 shows the $z$ profile obtained with the MT-scheme (on the bottom) and the $z$ profile obtained with the homogeneous system (on the top) at various instants. We notice that with the MT-scheme phase change occurs and the vapor bubble liquefies while for the homogenous off-equilibrium system (1) the bubble is just compressed.

## V. Conclusion

We have presented in this paper two variants of a convection-relaxation numerical method for computing phase change phenomena within the framework of a simple two-phase compressible isothermal flows model. The mass transfer effects are treated either by a source term integration step, either by an equilibrium states projection step. Numerical tests have been presented which show the ability of the method to capture solutions involving dynamical phase change processes remaining close to the thermodynamical equilibrium. As the equilibrium system is not strictly hyperbolic, stability and convergence behavior of the approximate solutions are a critical issue. Nevertheless, both space convergence and source term integration in the instantaneous relaxation limit and have been successfully challenged in the numerical tests.

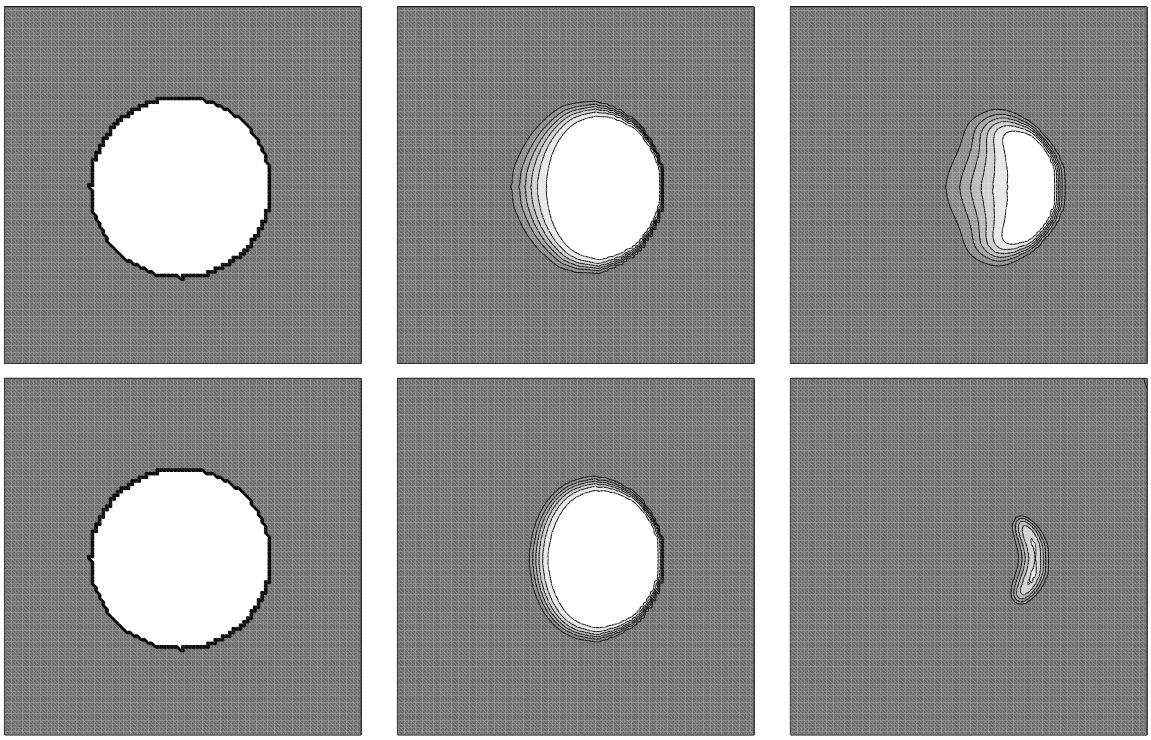


**Figure 4. $z$ profile obtained with the MT-scheme on the bottom and $z$ profile obtained with the homogeneous system on the top; $t$ is varies from $t = 0$ s to $t = 1.08$ ms.**

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